

The zeta function of a finite category and the series Euler characteristic

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Abstract

We prove that a certain conjecture holds true and the conjecture states a relationship between the zeta function of a finite category and the Euler characteristic of a finite category.

1 Introduction

In [NogA], the zeta function of a finite category was defined and one conjecture was proposed. The zeta function of a finite category I is the formal power series defined by

$$\zeta_I(z) = \exp \left(\sum_{m=1}^{\infty} \frac{\#N_m(I)}{m} z^m \right)$$

where

$$N_m(I) = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_m} x_m) \text{ in } I \}.$$

The conjecture states a relationship between the zeta function of a finite category and the Euler characteristic of a finite category, called *series Euler characteristic* [BL08].

Conjecture 1.1. Suppose I is a finite category which has series Euler characteristic. Then, we have

(C1) the zeta function of I is a finite product of the following form

$$\zeta_I(z) = \prod \frac{1}{(1 - \alpha_i z)^{\beta_i}} \exp \left(\sum \frac{\gamma_j z^j}{j(1 - \delta_j z)^j} \right)$$

for some complex numbers $\alpha_i, \beta_i, \gamma_j, \delta_j$.

(C2) $\sum \beta_i$ is the number of objects of I .

(C3) each α_i is an eigen value of A_I . Hence, α_i is an algebraic integer.

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$$(C4) \quad \sum \frac{\beta_i}{\alpha_i} + \sum (-1)^j \frac{\gamma_j}{\delta_j^{j+1}} = \chi_\Sigma(I).$$

It was verified this conjecture holds true under certain additional conditions in [NogA] and [NogB].

In [NogA], it was verified the conjecture holds true in concrete cases, that is, when a finite category is a groupoid, an acyclic category and has two objects and so on. An *acyclic* category is a small category in which all endomorphisms and isomorphisms are identity morphisms. In [NogB], it was verified the conjecture holds true when a finite category has Möbius inversion. A finite category I has *Möbius inversion* if its adjacency matrix A_I has an inverse matrix where A_I is an $N \times N$ -matrix whose (i, j) -entry is the number of morphisms of I from x_i to x_j when the set of objects of I is

$$\text{Ob}(I) = \{x_1, x_2, \dots, x_N\}$$

(see [Lei08] and [Lei]). In the sense of Leinster, this is called *coarse Möbius inversion* [Lei]. The class of finite categories which has coarse Möbius inversion is large and very important to consider the Euler characteristic of a finite category. Euler characteristic for categories is defined by various ways, *the series Euler characteristic* χ_Σ [BL08], *the L^2 -Euler characteristic* $\chi^{(2)}$ [FLS11], *the extended L^2 -Euler characteristic* $\chi_{\text{ex}}^{(2)}$ [Nog], *the Euler characteristic of an \mathbb{N} -filtered acyclic category* χ_{fl} [Nog11] and so on. If a finite category I has the coarse Möbius inversion, then I has Leinster's Euler characteristic and series Euler characteristic and they coincide, $\chi_L(I) = \chi_\Sigma(I)$. A finite acyclic category A has the coarse Möbius inversion and all of the Euler characteristic above for A coincide.

In this paper, we prove the conjecture holds true without any additional conditions. The following is our main theorem.

Main Theorem. Suppose I has series Euler characteristic and

$$\deg(|E - A_I z|) = N - r$$

and

$$\deg(\text{sum}(\text{adj}(E - A_I z))) = N - 1 - s$$

and the polynomial $|E - A_I z|$ is factored to the following form

$$|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \dots (z - \theta_n)^{e_n}$$

where each $e_i \geq 1$ and $\theta_i \neq \theta_j$ if $i \neq j$. Then the rational function

$$\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|}$$

has a partial fraction decomposition to the following form

$$\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}$$

for some complex numbers $A_{k,j}$. Moreover,

1. Then the zeta function of I is

$$\zeta_I(z) = \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \exp \left(\frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{z^j}{j \left(1 - \frac{1}{\theta_k} z\right)^j} \left(\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i+1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1} \right) \right)$$

$$2. \sum_{k=1}^n -\frac{A_{k,1}}{d_{N-r}} = N$$

3. Each $\frac{1}{\theta_k}$ is an eigen value of A_I . In particular, $\frac{1}{\theta_k}$ is an algebraic integer

4.

$$\sum_{k=1}^n \frac{-\frac{A_{k,1}}{d_{N-r}}}{\frac{1}{\theta_k}} + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} (-1)^j \frac{\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1}}{\left(\frac{1}{\theta_k}\right)^{j+1}} = \chi_\Sigma(I).$$

If we do not assume the condition that I has series Euler characteristic, the part 1 is given by the following.

Theorem 1.2 (Theorem 3.1). *Let I be a finite category. Suppose the polynomial $|E - A_I z|$ is factored to the following form:*

$$|E - A_I z| = d_{N-r} (z - \theta_1)^{e_1} \dots (z - \theta_n)^{e_n}$$

where $1 \leq r \leq N-1$ each $e_i \geq 1$ and $\theta_i \neq \theta_j$ if $i \neq j$. Suppose

$$\text{sum}(\text{adj}(E - A_I z) A_I) = q(z) |E - A_I z| + r(z)$$

where

$$\deg(r(z)) < \deg |E - A_I z|$$

and $\frac{r(z)}{|E - A_I z|}$ has a partial fraction decomposition to the following form

$$\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}.$$

Then the zeta function of I is

$$\zeta_I(z) = \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \exp \left(Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{z^j}{j \left(1 - \frac{1}{\theta_k} z\right)^j} \left(\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1} \right) \right)$$

where $Q(z) = \int q(z) dz$ is a polynomial whose constant term is 0.

It is very important to study about behavior of singular points and zeros of a zeta function. By the following corollary, the problem is reduced to investigate properties of roots of $|E - A_I z|$.

Corollary 1.3. *Let I be a finite category. A complex number z is a singular point or zero of ζ_I if and only if z is a root of $|E - A_I z|$.*

This paper is organized as follows.

In section 2, we prove some lemmas for a proof of our main theorem.

In section 3, we prove our main theorem.

2 Preparations for our main theorem

2.1 Notation

Throughout this paper, we will use the following notations.

1. We mean I is a finite category which has N -objects.
2. The three polynomials $|E - A_I z|$, $\text{sum}(\text{adj}(E - A_I z))$ and $\text{sum}(\text{adj}(E - A_I z)A_I)$ which will be often used later are expressed by the following form

$$|E - A_I z| = d_0 + d_1 z + \cdots + d_N z^N,$$

$$\text{sum}(\text{adj}(E - A_I z)) = k_0 + k_1 z + \cdots + k_{N-1} z^{N-1}$$

and

$$\text{sum}(\text{adj}(E - A_I z)A_I) = m_0 + m_1 z + \cdots + m_{N-1} z^{N-1}.$$

By Lemma 2.2 of [NogB], the degree of the third polynomial is less than or equal to $N - 1$. The coefficients d_0, d_1 and d_N are $1, (-1)^N \text{tr}(A_I)$ and $(-1)^N |A_I|$, respectively. Hence, the degree of $|E - A_I z|$ is larger or equal to 1 if I is not an empty category since $\text{tr}(A_I) \geq N$.

2.2 Some lemmas

In this subsection, we investigate the three polynomials above.

Lemma 2.1. *The degree of $|E - A_I z|$ is $N - r$ if and only if $|A_I - Ez|$ can be divided by z^r , but can not be divided by z^{r+1} .*

Proof. We have

$$|A_I - Ez| = (-1)^N (d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N).$$

Indeed, if we write

$$|A_I - Ez| = a_0 + a_1 z + \cdots + a_N z^N,$$

then we have

$$\begin{aligned} |E - A_I z| &= (-1)^N z^N \left| A_I - E \frac{1}{z} \right| \\ &= (-1)^N z^N \left(a_0 + a_1 \frac{1}{z} + \cdots + a_N \frac{1}{z^N} \right) \\ &= (-1)^N (a_0 z^N + a_1 z^{N-1} + \cdots + a_N) \\ &= d_0 + d_1 z + \cdots + d_N z^N. \end{aligned}$$

Hence, we have $a_0 = (-1)^N d_N, a_1 = (-1)^N d_{N-1}, \dots, a_N = (-1)^N d_0$.

Suppose $\deg(|E - A_I z|) = N - r$. Then, $d_N = d_{N-1} = \dots = d_{N-r+1} = 0$, but $d_{N-r} \neq 0$. Hence, we have

$$|A - Ez| = (-1)^N d_0 z^N + \dots + (-1)^N d_{N-r} z^r.$$

So $|A_I - Ez|$ can be divided by z^r , but can not be divided by z^{r+1} .

Conversely, if the polynomial $|A_I - Ez|$ can be divided by z^r , but can not be divided by z^{r+1} , then $d_N = d_{N-1} = \dots = d_{N-r+1} = 0$ and $d_{N-r} \neq 0$. Hence, $\deg(|E - A_I z|) = N - r$. \square

Lemma 2.2. *The degree of $\text{sum}(\text{adj}(E - A_I z))$ is $N - 1 - s$ if and only if $\text{sum}(\text{adj}(E - A_I z))$ can be divided by z^s , but can not be divided by z^{s+1} .*

Proof. We have

$$\text{sum}(\text{adj}(A_I - Ez)) = (-1)^{N-1} (k_0 z^{N-1} + k_1 z^{N-2} + \dots + k_{N-1}).$$

Indeed, if we write

$$\text{sum}(\text{adj}(A_I - Ez)) = b_0 + b_1 z + \dots + b_{N-1} z^{N-1},$$

then we have

$$\begin{aligned} \text{sum}(\text{adj}(E - A_I z)) &= (-z)^{N-1} \text{sum} \left(\text{adj} \left(A_I - E \frac{1}{z} \right) \right) \\ &= (-z)^{N-1} \left(b_0 + b_1 \frac{1}{z} + \dots + b_{N-1} \frac{1}{z^{N-1}} \right) \\ &= (-1)^{N-1} b_0 z^{N-1} + \dots + (-1)^{N-1} b_{N-1} \\ &= k_0 + k_1 z + \dots + k_{N-1} z^{N-1}. \end{aligned}$$

Hence, we have $b_0 = (-1)^{N-1} k_{N-1}, b_1 = (-1)^{N-1} k_{N-2}, \dots, b_{N-1} = (-1)^{N-1} k_0$.

Suppose $\deg(\text{sum}(\text{adj}(E - A_I z))) = N - 1 - s$. Then, $k_{N-1} = k_{N-2} = \dots = k_{N-s} = 0$, but $k_{N-s-1} \neq 0$. Hence, we have

$$\text{sum}(\text{adj}(E - A_I z)) = (-1)^{N-1} k_0 z^{N-1} + \dots + (-1)^{N-1} k_{N-1-s} z^s.$$

So $\text{sum}(\text{adj}(E - A_I z))$ can be divided by z^s , but can not be divided by z^{s+1} .

Conversely, if the polynomial $\text{sum}(\text{adj}(E - A_I z))$ can be divided by z^s , but can not be divided by z^{s+1} , then $k_{N-1} = k_{N-2} = \dots = k_{N-s} = 0$ and $k_{N-1-s} \neq 0$. Hence, $\deg(\text{sum}(\text{adj}(E - A_I z))) = N - 1 - s$. \square

Lemma 2.3. *Suppose the degree of $|E - A_I z|$ is $N - r$ and the degree of $\text{sum}(\text{adj}(E - A_I z))$ is $N - 1 - s$. Then, I has series Euler characteristic if and only if $s \geq r$. In this case, we have*

$$\chi_\Sigma(I) = \begin{cases} 0 & \text{if } s > r \\ -\frac{k_{N-1-s}}{d_{N-r}} & \text{if } s = r \end{cases}.$$

Proof. The finite category I has series Euler characteristic if and only if the rational function

$$\frac{\text{sum}(\text{adj}(E - (A_I - E)t))}{|E - (A_I - E)t|}$$

can be substituted -1 to t if and only if the rational function

$$\frac{\text{sum}(\text{adj}(A_I - Ez))}{|A_I - Ez|}$$

can be substituted 0 to z (page 45 of [BL08]). Lemma 2.1 and Lemma 2.2 imply

$$\frac{\text{sum}(\text{adj}(A_I - Ez))}{|A_I - Ez|} = \frac{z^s h(z)}{z^r g(z)}$$

for some polynomials $g(z)$ and $h(z)$ of $\mathbb{Z}[z]$ such that $g(z)$ and $h(z)$ can not be divided by z . Hence, the rational function

$$\frac{\text{sum}(\text{adj}(A_I - Ez))}{|A_I - Ez|}$$

can be substituted 0 to z if and only if $s \geq r$. So the first claim is proved.

Suppose I has series Euler characteristic. Then, we have $s \geq r$. If $s > r$, then it is clear $\chi_\Sigma(I) = 0$. If $s = r$, then we have

$$\begin{aligned} \frac{\text{sum}(\text{adj}(A_I - Ez))}{|A_I - Ez|} &= \frac{(-1)^{N-1}(k_0 z^{N-1} + k_1 z^{N-2} + \cdots + k_{N-1-s} z^s)}{(-1)^N(d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-r} z^r)} \\ &= -\frac{k_0 z^{N-1-s} + \cdots + k_{N-1-s}}{d_0 z^{N-r} + \cdots + d_{N-r}}. \end{aligned}$$

Hence, we obtain $\chi_\Sigma(I) = -\frac{k_{N-1-s}}{d_{N-r}}$. □

Lemma 2.4. *If I has series Euler characteristic, then we have*

$$\deg(\text{sum}(\text{adj}(E - A_I z)A_I)) = \deg(|E - A_I z|) - 1.$$

Proof. Lemma 2.2 of [NogB] implies

$$\text{sum}(\text{adj}(E - A_I z)A_I) = \frac{1}{z} \left(\text{sum}(\text{adj}(E - A_I z)) - N|E - A_I z| \right).$$

Note that the polynomial

$$\text{sum}(\text{adj}(E - A_I z)) - N|E - A_I z|$$

has no constant term since $k_0 = N$ and $d_0 = 1$. Hence, we have

$$\deg(\text{sum}(\text{adj}(E - A_I z)A_I)) = \deg\left(\text{sum}(\text{adj}(E - A_I z)) - N|E - A_I z|\right) - 1.$$

Since I has series Euler characteristic, Lemma 2.3 implies $s \geq r$. Hence, we have the inequality

$$\deg(\text{sum}(\text{adj}(E - A_I z))) = N - 1 - s < N - r = \deg(|E - A_I z|).$$

So we obtain

$$\deg(\text{sum}(\text{adj}(E - A_I z)A_I)) = \deg(|E - A_I z|) - 1.$$

□

Lemma 2.5. *If I has series Euler characteristic and $\deg(|E - A_I z|) = N - r$ and*

$$\deg(\text{sum}(\text{adj}(E - A_I z))) = N - 1 - s,$$

then for the polynomial

$$\text{sum}(\text{adj}(E - A_I z)A_I) = m_0 + m_1 z + \cdots + m_{N-1-r} z^{N-1-r},$$

we have $m_{N-1-r} = -Nd_{N-r}$ and

$$m_{N-2-r} = \begin{cases} -Nd_{N-1-r} & \text{if } s > r \\ -Nd_{N-1-r} + k_{N-1-r} & \text{if } s = r. \end{cases}$$

Proof. Lemma 2.2 of [NogB] implies

$$\begin{aligned} \text{sum}(\text{adj}(E - A_I z)A_I) &= \text{sum}(\text{adj}(E - A_I z)A_I) = m_0 + m_1 z \\ &\quad + \cdots + m_{N-1-r} z^{N-1-r} \\ &= \frac{1}{z} \left(\text{sum}(\text{adj}(E - A_I z)) - N|E - A_I z| \right) \\ &= \frac{1}{z} \left(k_0 + k_1 z + \cdots + k_{N-1-s} z^{N-1-s} \right. \\ &\quad \left. - N(d_0 + d_1 z + \cdots + d_{N-r} z^{N-r}) \right) \\ &= (k_1 - Nd_1) + (k_2 - Nd_2)z + \cdots \\ &\quad + (k_{N-1-s} - Nd_{N-1-s})z^{N-2-s} - Nd_{N-s} z^{N-1-s} \\ &\quad + \cdots - Nd_{N-r} z^{N-1-r}. \end{aligned}$$

Since I has series Euler characteristic, Lemma 2.4 implies $s \geq r$. Hence,

$$N - 1 - s < N - r,$$

so that $m_{N-1-r} = -Nd_{N-r}$.

If $s > r$, then $N - 1 - s < N - 1 - r$, so that $m_{N-2-r} = -Nd_{N-1-r}$.

If $s = r$, then $m_{N-2-r} = -Nd_{N-1-r} + k_{N-1-r}$. □

3 A proof of main theorem

Theorem 3.1. *Let I be a finite category. Suppose the polynomial $|E - A_I z|$ is factored to the following form:*

$$|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \cdots (z - \theta_n)^{e_n}$$

where $1 \leq r \leq N - 1$ each $e_i \geq 1$ and $\theta_i \neq \theta_j$ if $i \neq j$. Suppose

$$\text{sum}(\text{adj}(E - A_I z)A_I) = q(z)|E - A_I z| + r(z)$$

where

$$\deg(r(z)) < \deg|E - A_I z|$$

and $\frac{r(z)}{|E - A_I z|}$ has a partial fraction decomposition to the following form

$$\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}.$$

Then the zeta function of I is

$$\begin{aligned} \zeta_I(z) = & \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \exp \left(Q(z) + \right. \\ & \left. \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{z^j}{j \left(1 - \frac{1}{\theta_k} z\right)^j} \left(\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1} \right) \right) \end{aligned}$$

where $Q(z) = \int q(z) dz$ is a polynomial whose constant term is 0.

Proof. Since

$$\deg(r(z)) < \deg |E - A_I z|,$$

we can have a partial fraction decomposition of the following form

$$\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}$$

for some complex numbers $A_{k,j}$. Hence, we have

$$\begin{aligned} \frac{\text{sum}(\text{adj}(E - A_I z) A_I)}{|E - A_I z|} &= q(z) + \frac{r(z)}{|E - A_I z|} \\ &= q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}. \end{aligned}$$

Proposition 2.1 of [NogB] implies

$$\begin{aligned}
\zeta_I(z) &= \exp \left(\int q(z) dz + \int \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j} dz \right) \\
&= \exp \left(\int q(z) dz + \frac{1}{d_{N-r}} \int \sum_{k=1}^n \frac{A_{k,1}}{(z - \theta_k)} dz + \right. \\
&\quad \left. \frac{1}{d_{N-r}} \int \sum_{k=1}^n \sum_{j=2}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j} dz \right) \\
&= \exp \left(Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^n A_{k,1} \log(z - \theta_k) + \right. \\
&\quad \left. \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=2}^{e_k} -\frac{A_{k,j}}{(j-1)} \frac{1}{(z - \theta_k)^{j-1}} + C \right) \\
&= \prod_{k=1}^n \frac{1}{(z - \theta_k)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \\
&\quad \exp \left(Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} -\frac{A_{k,j+1}}{j} \frac{1}{(z - \theta_k)^j} \right) \exp C \\
&= \prod_{k=1}^n \frac{1}{(-\theta_k)^{-\frac{A_{k,1}}{d_{N-r}}} \left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \\
&\quad \exp \left(Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} -\frac{A_{k,j+1}}{j} \frac{1}{(z - \theta_k)^j} \right) C' \\
&= \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \\
&\quad \exp \left(Q(z) + \frac{-1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{A_{k,j+1}}{j} \frac{1}{(z - \theta_k)^j} \right) C''
\end{aligned}$$

where we did and will replace the constant term as C, C' and $C'' \dots$. Lemma 2.7 of [NogB] implies

$$\begin{aligned}
\zeta_I(z) &= \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \\
&\quad \exp \left(Q(z) + \frac{-1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{A_{k,j+1}}{j} \sum_{i=1}^j \frac{\binom{j}{i} \left(-\frac{1}{\theta_k}\right)^j (-z)^i}{(z - \theta_k)^i} \right) C'''
\end{aligned}$$

Here, we use the boundary condition $\zeta_I(0) = 1$. This condition is directly implied by the definition of the zeta function. Hence, we obtain $C''' = 1$. By

exchanging \sum_i and \sum_j , we have

$$\zeta_I(z) = \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \exp \left(Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{z^j}{j \left(1 - \frac{1}{\theta_k} z\right)^j} \left(\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i+1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1} \right) \right)$$

Hence, we obtain the result. \square

It is very important to study about behavior of singular points and zeros of a zeta function. By the following corollary, the problem is reduced to investigate properties of roots of $|E - A_I z|$.

Corollary 3.2. *Let I be a finite category. A complex number z is a singular point or a zero of ζ_I if and only if z is a root of $|E - A_I z|$.*

Proof. Theorem 3.1 directly implies all of the singular points and zeros are roots of $|E - A_I z|$. Conversely, suppose z is a root of $|E - A_I z|$ but z is not a singular point and a zero. Then, $z = \theta_\ell$ for some ℓ . The index $-\frac{A_{\ell,1}}{d_{N-r}}$ must be 0. Namely, we have $A_{\ell,1} = 0$. For $j = e_\ell - 1$,

$$\sum_{i=e_\ell-1}^{e_\ell-1} -A_{\ell,i+1} \binom{i-1}{j-1} \left(-\frac{1}{\theta_\ell}\right)^{i-(e_\ell-1)}$$

must be 0 since $\zeta_I(z)$ is defined. Hence, we have $A_{\ell,e_\ell} = 0$. As this, we can show each $A_{\ell,j} = 0$ by the descent from $j = e_\ell - 1$. Hence, we have

$$\begin{aligned} \frac{r(z)}{|E - A_I z|} &= \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_\ell} \frac{A_{k,j}}{(z - \theta_\ell)^j} \\ &= \frac{1}{d_{N-r}} \sum_{k=1, k \neq \ell}^n \sum_{j=1}^{e_\ell} \frac{A_{k,j}}{(z - \theta_\ell)^j} \end{aligned}$$

Hence, we obtain

$$\begin{aligned} |E - A_I z| &= d_{N-r} (z - \theta_1)^{e_1} \dots (z - \theta_n)^{e_n} \\ &= d_{N-r} (z - \theta_1)^{e_1} \dots \\ &\quad (z - \theta_{\ell-1})^{e_{\ell-1}} (z - \theta_{\ell+1})^{e_{\ell+1}} \dots (z - \theta_n)^{e_n} \end{aligned}$$

The polynomial $|E - A_I z|$ has two different degrees since each $e_k \geq 1$. This contradiction implies $z = \theta_\ell$ is a singular point or a zero of ζ_I . \square

Theorem 3.3. *Suppose I has series Euler characteristic and*

$$\deg(|E - A_I z|) = N - r$$

and

$$\deg(\text{sum}(\text{adj}(E - A_I z))) = N - 1 - s$$

and the polynomial $|E - A_I z|$ is factored to the following form

$$|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \dots (z - \theta_n)^{e_n}$$

where each $e_i \geq 1$ and $\theta_i \neq \theta_j$ if $i \neq j$. Then the rational function

$$\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|}$$

has a partial fraction decomposition to the following form

$$\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}$$

for some complex numbers $A_{k,j}$. Moreover,

1. the zeta function of I is

$$\zeta_I(z) = \prod_{k=1}^n \frac{1}{\left(1 - \frac{1}{\theta_k} z\right)^{-\frac{A_{k,1}}{d_{N-r}}}} \times \exp \left(\frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} \frac{z^j}{j \left(1 - \frac{1}{\theta_k} z\right)^j} \left(\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1} \right) \right)$$

2. the sum of all the indexes are the number of objects of I , that is,

$$\sum_{k=1}^n -\frac{A_{k,1}}{d_{N-r}} = N$$

3. Each $\frac{1}{\theta_k}$ is an eigen value of A_I . In particular, $\frac{1}{\theta_k}$ is an algebraic integer

4.

$$\sum_{k=1}^n \frac{-\frac{A_{k,1}}{d_{N-r}}}{\frac{1}{\theta_k}} + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} (-1)^j \frac{\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1}}{\left(\frac{1}{\theta_k}\right)^{j+1}} = \chi_\Sigma(I). \quad (1)$$

We give a simple interpretation of the part 4. Put $\alpha_k = \frac{1}{\theta_k}$, $\beta_{k,0} = -\frac{A_{k,1}}{d_{N-r}}$ and

$$\beta_{k,j} = \sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1}.$$

Then, the equation (1) is

$$\sum_{k=1}^n \sum_{j=0}^{e_k-1} (-1)^j \frac{\beta_{k,j}}{\alpha_k^{j+1}} = \chi_\Sigma(I).$$

This theorem claims that this alternating sum is always a rational number and it is the series Euler characteristic $\chi_\Sigma(I)$ of I .

Proof of Theorem 3.3. Lemma 2.4 implies

$$\deg(\text{sum}(\text{adj}(E - A_I z)A_I)) < \deg(|E - A_I z|).$$

Hence, we have a partial fraction decomposition

$$\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}$$

for some complex numbers $A_{k,j}$.

The part 1 is directly implied by Theorem 3.1 as $Q(z) = 0$.

Next we show the part 2. We observe the numerators of both sides

$$\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}.$$

For the right hand side, when it is transformed to the left hand side by a reduction to common denominator, the coefficient of z^{N-1-r} of the numerator is $\sum_{k=1}^n A_{k,1}$. Lemma 2.5 implies $\sum_{k=1}^n A_{k,1} = m_{N-1-r} = d_{N-r}$. Thus, we obtain

$$\sum_{k=1}^n \frac{A_{k,1}}{d_{N-r}} = N.$$

We show the part 3. Since each θ_k is a root of the polynomial $|E - A_I z|$, we obtain

$$\begin{aligned} |E - A_I \theta_k| &= 0 \\ (\theta_k)^N \left| E \frac{1}{\theta_k} - A_I \right| &= 0. \end{aligned}$$

Hence, $\frac{1}{\theta_k}$ is an eigen value of A_I . Note that $\theta_k \neq 0$. Moreover, since $|E\lambda - A_I|$ is a monic polynomial with coefficients in \mathbb{Z} , $\frac{1}{\theta_k}$ is an algebraic integer.

Finally, we show the part 4. The equation (1) is

$$\begin{aligned}
(1) &= \sum_{k=1}^n \frac{-\frac{A_{k,1}}{d_{N-r}}}{\frac{1}{\theta_k}} \\
&\quad + \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k-1} (-1)^j \frac{\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left(\frac{1}{\theta_k}\right)^{i+j} A_{k,i+1}}{\left(\frac{1}{\theta_k}\right)^{j+1}} \\
&= \sum_{k=1}^n \left(-\frac{\theta_k A_{k,1}}{d_{N-r}} \right. \\
&\quad \left. + \frac{1}{d_{N-r}} \sum_{j=1}^{e_k-1} \sum_{i=j}^{e_k-1} (-1)^j \left(-\frac{1}{\theta_k}\right)^{i-1} \binom{i-1}{j-1} A_{k,i+1} \right) \\
&= \sum_{k=1}^n \left(-\frac{\theta_k A_{k,1}}{d_{N-r}} \right. \\
&\quad \left. + \frac{1}{d_{N-r}} \sum_{i=1}^{e_k-1} \left(-\frac{1}{\theta_k}\right)^{i-1} A_{k,i+1} \left(\sum_{j=1}^i (-1)^j \binom{i-1}{j-1} \right) \right) \\
&= \frac{1}{d_{N-r}} \left(\sum_{k=1}^n (-\theta_k A_{k,1} - A_{k,2}) \right).
\end{aligned}$$

So it is enough to show

$$\frac{1}{d_{N-r}} \left(\sum_{k=1}^n -\theta_k A_{k,1} - A_{k,2} \right) = \chi_{\Sigma}(I). \quad (2)$$

By comparison of the numerators of both sides

$$\frac{\text{sum}(\text{adj}(E - A_I z) A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^n \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j},$$

we have

$$m_{N-2-r} = \sum_{k=1}^n A_{k,2} - \sum_{k=1}^n A_{k,1} (\theta_1 e_1 + \dots + \theta_k (e_k - 1) + \dots + \theta_n e_n).$$

Hence, the left hand side of (2) is

$$\begin{aligned}
\frac{1}{d_{N-r}} \left(\sum_{k=1}^n -\theta_k A_{k,1} - A_{k,2} \right) &= \frac{1}{d_{N-r}} \left(\sum_{k=1}^n -\theta_k A_{k,1} - m_{N-2-r} \right. \\
&\quad \left. - A_{k,1} (\theta_1 e_1 + \theta_k (e_k - 1) + \dots + \theta_n e_n) \right) \\
&= \frac{1}{d_{N-r}} \left(- \left(\sum_{k=1}^n A_{k,1} \right) \left(\sum_{k=1}^n \theta_k e_k \right) \right. \\
&\quad \left. - m_{N-2-r} \right). \quad (3)
\end{aligned}$$

We have

$$\begin{aligned}
|E - A_I z| &= d_0 + d_1 z + \cdots + d_{N-r} z^{N-r} \\
&= d_{N-r} (z - \theta_1)^{e_1} \cdots (z - \theta_n)^{e_n} \\
&= d_{N-r} \left(z^{N-r} - \left(\sum_{k=1}^n \theta_k e_k \right) z^{N-1-r} + \cdots \right).
\end{aligned}$$

Hence, we obtain $-d_{N-r}(\sum_{k=1}^n \theta_k e_k) = d_{N-1-r}$, so that

$$-\sum_{k=1}^n \theta_k e_k = \frac{d_{N-1-r}}{d_{N-r}}.$$

We have already seen $\sum_{k=1}^n A_{k,1} = -Nd_{N-r}$. Therefore, the equation (3) is

$$\begin{aligned}
\frac{1}{d_{N-r}} \left(- \left(\sum_{k=1}^n A_{k,1} \right) \left(\sum_{k=1}^n \theta_k e_k \right) - m_{N-2-r} \right) = \\
\frac{1}{d_{N-r}} \left(-Nd_{N-1-r} - m_{N-2-r} \right). \quad (4)
\end{aligned}$$

Here we have to consider two cases

$$\chi_{\Sigma}(I) = \begin{cases} 0 & \text{if } s > r \\ -\frac{k_{N-1-s}}{d_{N-r}} & \text{if } s = r \end{cases}$$

(see Lemma 2.3).

If $s > r$, Lemma 2.5 implies $m_{N-2-r} = -Nd_{N-1-r}$, so that the equation (4) is

$$\begin{aligned}
\frac{1}{d_{N-r}} \left(-Nd_{N-1-r} - m_{N-2-r} \right) &= \frac{1}{d_{N-r}} \left(-Nd_{N-1-r} + Nd_{N-1-r} \right) \\
&= 0 \\
&= \chi_{\Sigma}(I).
\end{aligned}$$

If $r = s$, Lemma 2.3 implies $m_{N-r-2} = k_{N-1-r} - Nd_{N-1-r}$. Hence, the equation (4) is

$$\begin{aligned}
\frac{1}{d_{N-r}} \left(-Nd_{N-1-r} - m_{N-2-r} \right) &= \frac{1}{d_{N-r}} \left(-Nd_{N-1-r} - k_{N-1-r} + Nd_{N-1-r} \right) \\
&= -\frac{k_{N-1-r}}{d_{N-r}} \\
&= \chi_{\Sigma}(I).
\end{aligned}$$

Hence, we obtain the results. \square

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